Applications of Legendre polynomials

Expanding a $1/r$ potential

The Legendre polynomials were first introduced in 1782 by Adrien-Marie Legendre[2] as the coefficients in the expansion of the Newtonian potential

$$
\frac{1}{|x - x'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \gamma}} = \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}} P_l(\cos \gamma),
$$

where $r$ and $r'$ are the lengths of the vectors $x$ and $x'$ respectively and $\gamma$ is the angle between those two vectors. The series converges when $r > r'$. The expression gives the gravitational potential associated to a point mass or the Coulomb potential associated to a point charge. The expansion using Legendre polynomials might be useful, for instance, when integrating this expression over a continuous mass or charge distribution.

Legendre polynomials occur in the solution of Laplace’s equation of the static potential, $\nabla^2 \Phi(x) = 0$, in a charge-free region of space, using the method of separation of variables, where the boundary conditions have axial symmetry (no dependence on an azimuthal angle). Where $z$ is the axis of symmetry and $\theta$ is the angle between the position of the observer and the $z$ axis (the zenith angle), the solution for the potential will be

$$
\Phi(r, \theta) = \sum_{l=0}^{\infty} \left( A_l r^l + B_l r^{-(l+1)} \right) P_l(\cos \theta).
$$

$A_l$ and $B_l$ are to be determined according to the boundary condition of each problem.[3]

They also appear when solving the Schrödinger equation in three dimensions for a central force.
Legendre polynomials in multipole expansions

Legendre polynomials are also useful in expanding functions of the form (this is the same as before, written a little differently):

\[
\frac{1}{\sqrt{1+\eta^2-2\eta x}} = \sum_{k=0}^{\infty} \eta^k P_k(x),
\]

which arise naturally in multipole expansions. The left-hand side of the equation is the generating function for the Legendre polynomials.

As an example, the electric potential \(\Phi(r, \theta)\) (in spherical coordinates) due to a point charge located on the \(z\)-axis at \(z = a\) (see diagram right) varies as

\[
\Phi(r, \theta) \propto \frac{1}{r} = \frac{1}{\sqrt{r^2 + a^2 - 2ar \cos \theta}}.
\]

If the radius \(r\) of the observation point \(P\) is greater than \(a\), the potential may be expanded in the Legendre polynomials

\[
\Phi(r, \theta) \propto \frac{1}{r} \sum_{k=0}^{\infty} \left( \frac{a}{r} \right)^k P_k(\cos \theta),
\]

where we have defined \(\eta = \frac{a}{r} < 1\) and \(x = \cos \theta\). This expansion is used to develop the normal multipole expansion.

Conversely, if the radius \(r\) of the observation point \(P\) is smaller than \(a\), the potential may still be expanded in the Legendre polynomials as above, but with \(a\) and \(r\) exchanged. This expansion is the basis of interior multipole expansion.

Legendre polynomials in trigonometry

The trigonometric functions \(\cos n \theta\), also denoted as the Chebyshev polynomials \(T_n(\cos \theta) \equiv \cos n \theta\), can also be multipole expanded by the Legendre polynomials \(P_n(\cos \theta)\). The first several orders are as follows:

\[
\begin{align*}
T_0(\cos \theta) &= 1 = P_0(\cos \theta), \\
T_1(\cos \theta) &= \cos \theta = P_1(\cos \theta), \\
T_2(\cos \theta) &= \cos 2\theta = \frac{1}{2} \left( 4P_2(\cos \theta) - P_0(\cos \theta) \right)
\end{align*}
\]

\[
\begin{align*}
T_3(\cos \theta) &= \cos 3\theta = \frac{1}{3} \left( 8P_3(\cos \theta) - 3P_1(\cos \theta) \right)
\end{align*}
\]

\[
\begin{align*}
T_4(\cos \theta) &= \cos 4\theta = \frac{1}{16} \left( 192P_4(\cos \theta) - 80P_2(\cos \theta) - 7P_0(\cos \theta) \right)
\end{align*}
\]

\[
\begin{align*}
T_5(\cos \theta) &= \cos 5\theta = \frac{1}{63} \left( 128P_5(\cos \theta) - 56P_3(\cos \theta) - 9P_1(\cos \theta) \right)
\end{align*}
\]

\[
\begin{align*}
T_6(\cos \theta) &= \cos 6\theta = \frac{1}{18} \left( 2560P_6(\cos \theta) - 1152P_4(\cos \theta) - 220P_2(\cos \theta) - 33P_0(\cos \theta) \right)
\end{align*}
\]

Another property is the expression for \(\sin (n + 1)\theta\), which is

\[
\frac{\sin(n+1)\theta}{\sin \theta} = \sum_{l=0}^{n} P_l(\cos \theta) P_{n-l}(\cos \theta).
\]

Legendre polynomials in recurrent neural networks

A recurrent neural network that contains a \(d\)-dimensional memory vector, \(\mathbf{m} \in \mathbb{R}^d\), can be optimized such that its neural activities obey the linear time-invariant system given by the following state-space representation:

\[
\dot{\mathbf{m}}(t) = A \mathbf{m}(t) + B u(t)
\]

\[
A = [a_{ij}] \in \mathbb{R}^{d \times d}, \quad a_{ij} = (2i + 1) \begin{cases} -1 & i < j \\ (-1)^{i-j+1} & i \geq j \end{cases},
\]

\[
B = [b_i] \in \mathbb{R}^{d \times 1}, \quad b_i = (2i + 1)(-1)^i.
\]

In this case, the sliding window of \(u\) across the past \(\theta\) units of time is best approximated by a linear combination of the first \(d\) shifted Legendre polynomials, weighted together by the elements of \(\mathbf{m}\) at time \(t\):

\[
u(t - \theta') \approx \sum_{t=0}^{d-1} \hat{P}_t \left( \frac{\theta'}{\theta} \right) m_t(t), \quad 0 \leq \theta' \leq \theta.
\]

When combined with deep learning methods, these networks can be trained to outperform long short-term memory units and related architectures, while using fewer computational resources.\[4\]